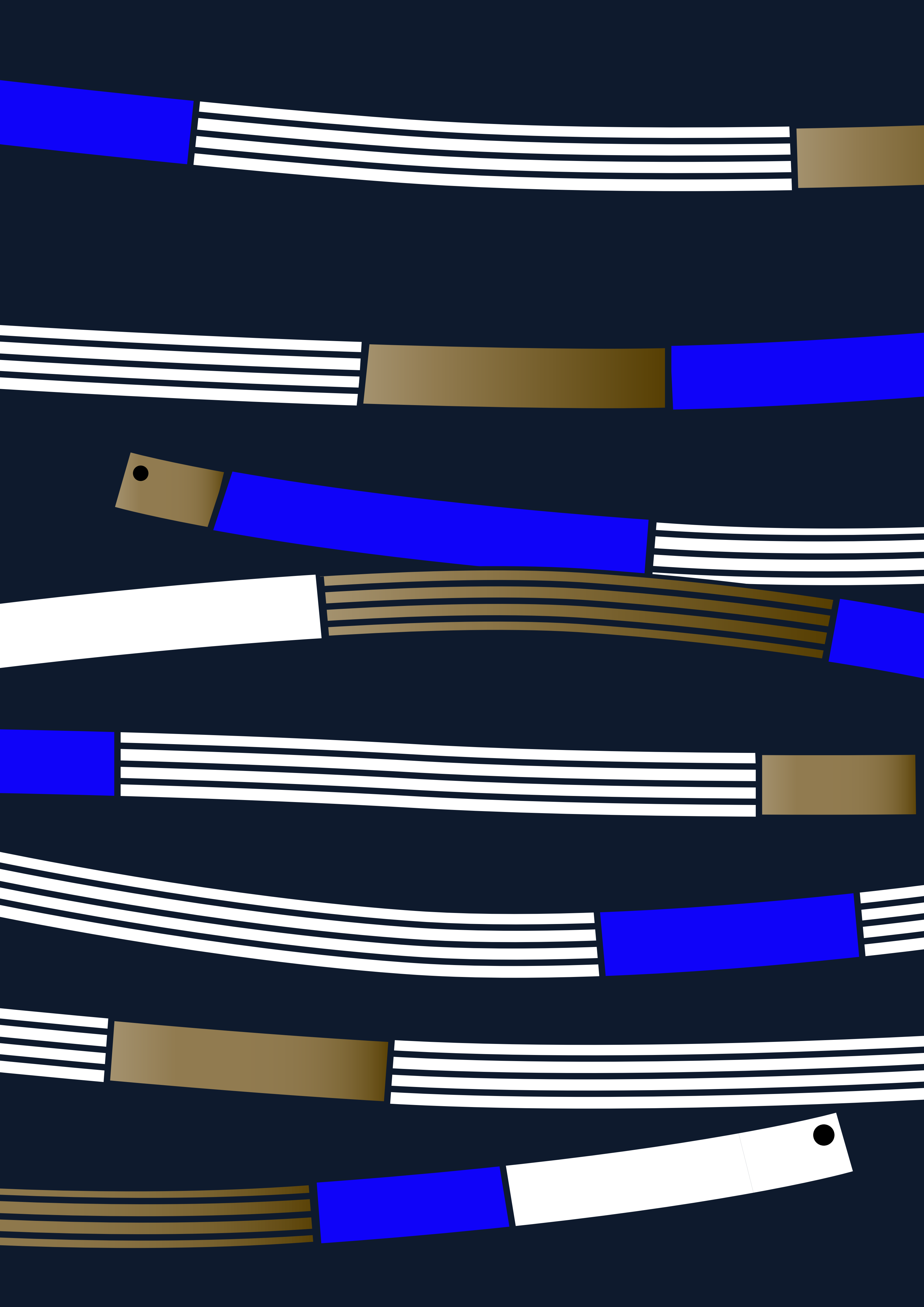

Linear Transformation

Let U and V be two vector spaces over field F. A mapping $T: U \to V$ is called a linear transformation or a homomorphism if it satisfies the following two conditions

i) $T(x+y) = T(x) + T(y)$ for $x, y \in V$ & $\alpha \in F$

ii) $T(\alpha x) = \alpha T(x)$

Or

$T: U \to V$ is called a linear transformation

if $\boxed{T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)}$

for $\alpha, \beta \in F$, $x, y \in V$

Ex: Show that the mapping $T: R^2 \to R^2$ defined as $T(\alpha, \beta) = (\alpha, 0)$ is a linear transformation.

Soln: Let $x = (\alpha_1, \beta_1)$ & $y = (\alpha_2, \beta_2) \in R^2$ & $\alpha, \beta \in R$

Now,
$$\alpha x + \beta y = \alpha(\alpha_1, \beta_1) + \beta(\alpha_2, \beta_2)$$
$$= (\alpha\alpha_1, \alpha\beta_1) + (\beta\alpha_2, \beta\beta_2)$$
$$= (\alpha\alpha_1 + \beta\alpha_2, \alpha\beta_1 + \beta\beta_2)$$

Again,
$$T(\alpha x + \beta y) = T(\alpha\alpha_1 + \beta\alpha_2, \alpha\beta_1 + \beta\beta_2)$$
$$= (\alpha\alpha_1 + \beta\alpha_2, 0)$$
$$= (\alpha\alpha_1, 0) + (\beta\alpha_2, 0)$$
$$= \alpha(\alpha_1, 0) + \beta(\alpha_2, 0)$$
$$= \alpha T(\alpha_1, \beta_1) + \beta T(\alpha_2, \beta_2)$$

$$T(\alpha x+\beta y) = \alpha T(x) + \beta T(y)$$

Hence, T is a L.T.

Q. Show that the mapping $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined $T(x_1, x_2) = (x_1+x_2, x_1-x_2, x_2)$ is a L.T.

Sol: Let $x = (x_1, x_2)$ & $y = (y_1, y_2) \in \mathbb{R}^2$ & $\alpha, \beta \in \mathbb{R}$

Now
$$\alpha x+\beta y = \alpha(x_1, x_2) + \beta(y_1, y_2)$$
$$= (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)$$
$$= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

Again,
$$T(\alpha x+\beta y) = T(\alpha x_1+\beta y_1, \alpha x_2+\beta y_2)$$
$$= (\alpha x_1+\beta y_1+\alpha x_2+\beta y_2, \alpha x_1+\beta y_1-\alpha x_2-\beta y_2, \alpha x_2+\beta y_2)$$
$$= (\alpha x_1+\alpha x_2, \alpha x_1-\alpha x_2, \alpha x_2) + (\beta y_1+\beta y_2, \beta y_1-\beta y_2, \beta y_2)$$
$$= \alpha(x_1+x_2, x_1-x_2, x_2) + \beta(y_1+y_2, y_1-y_2, y_2)$$
$$= \alpha T(x_1, x_2) + \beta T(y_1, y_2)$$
$$T(\alpha x+\beta y) = \alpha T(x) + \beta T(y)$$

Hence, T is a Linear transformation

Q. If the mapping $T: C \to C$ defined by $T(x+iy) = x$ is a L.T., where C is a vector space of complex no. over itself.

Sol: Let $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2 \in C$

$$\boxed{\begin{aligned} T(z_1) &= T(x_1+iy_1) \\ &= x_1 \\ \& \ T(z_2) &= T(x_2+iy_2) \\ &= x_2 \end{aligned}}$$

Now, $T(z_1+z_2) = T\{(x_1+iy_1) + (x_2+iy_2)\}$

$= T\{(x_1+x_2) + i(y_1+y_2)\}$

$= x_1 + x_2$

$= T(z_1) + T(z_2)$

Let $\alpha = 1-i \in C$ & $z = 1+i \in C$

Now, $\alpha z = (1-i)(1+i) = 1 - i^2 = 1+1 = 2$

$\therefore \underline{T(\alpha z)} = T(2) = T(2+i\cdot 0) = 2$

& $\alpha T(z) = (1-i)\,T(1+i)$

$= (1-i).1 = 1-i$

Thus $T(\alpha z) \neq \alpha T(z)$

Hence, T is not a L.T.

Kernel and Range of L.T.

Let $T: U \to V$ be a linear transformation. Then the kernel of T denoted by Ker T or $N(T)$ defined as

$$\boxed{\text{Ker } T = \{x \in V : T(x) = 0 \in V\}}$$

Ker T is also called the null space of T

Range: Let $T: V \to U$ is a L.T. The range of T denoted by $R(T)$ or $T(V)$ and defined by

$$\boxed{R(T) = \{T(x) : x \in V\}}$$

Isomorphism: A linear transformation $T: V \to U$ is called isomorphism if T is one one

i.e $T(x) = T(y) \Rightarrow x = y$; $x, y \in V$

Isomorphic: A vector $V(F)$ is said to be a isomorphic to a vector space $U(F)$ if there exists a mapping $T: V \to U$ such that

i) T is L.T.

ii) T is one one

iii) T is onto i.e for each $u \in U$ $\exists$ some $v \in V$ such that $T(v) = u$

Q. Let $T: V \longrightarrow U$ be a L.T. Then

i) $T(0) = 0$

ii) $T(-x) = -T(x)$

iii) $T(x-y) = T(x) - T(y)$ $\forall\, x, y \in V$

Sol: i) We have

$$0 + 0 = 0$$

$\Rightarrow T(0+0) = T(0)$

$\Rightarrow T(0) + T(0) = T(0) + 0$ $\because$ As T is L.T.

$\Rightarrow T(0) = 0$ (by L.C.L)

i) $T(x+y) = T(x) + T(y)$

ii) $T(\alpha x) = \alpha T(x)$

ii) We have

$$x+(-x)=0$$

$$\Rightarrow T\{x+(-x)\} = T(0)$$

$$\Rightarrow T(x)+T(-x) = 0 \qquad \because \text{As } T \text{ is L.T.}$$

$$\Rightarrow T(-x) = -T(x)$$

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iii) $T(x-y) = T\{x+(-y)\}$

$$= T(x)+T(-y)$$

$$= T(x)-T(+y) \qquad \text{(by ii)}$$

$$\therefore T(x-y) = T(x)-T(y)$$